Dynamical Systems Method for solving equations with non-smooth monotone operators *

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
E:mail: ramm@math.ksu.edu

Abstract

Consider an operator equation (*) $B(u) + \epsilon u = 0$ in a real Hilbert space, where $\epsilon > 0$ is a small constant. The DSM (dynamical systems method) for solving equation (*) consists of a construction of a Cauchy problem, which has the following properties: 1) it has a global solution for an arbitrary initial data,
2) this solution tends to a limit as time tends to infinity,
3) the limit solves the equation $B(u) = 0$.
Existence of the unique solution is proved by the DSM for equation (*) with monotone hemicontinuous operators B defined on all of H. If $\epsilon = 0$ and equation (**) $B(u) = 0$ is solvable, the DSM yields a solution to (**).

1 Introduction

In this paper a version of the DSM, dynamical systems method, is proposed for solving nonlinear operator equation of the form:

$$B(v) + \epsilon v = 0, \quad \epsilon = const > 0, \tag{1}$$

where the operator $B : H \to H$ is a nonlinear monotone map in a Hilbert space H, and, in contrast to our earlier work [2], this operator may be non-smooth.

The notions related to monotone operators, used in this paper, one finds, e.g., in [1]. The DSM is applied to solving operator equations in [2], where the nonlinear mapping B was assumed locally twice Fréchet differentiable. In this paper existence of the Fréchet derivative of $B(u)$ is not assumed. The novel

*Math subject classification: 34R30, 35R25, 35R30, 37C35, 37L05, 37N30, 47A52, 47J06, 65M30, 65N21 key words: dynamical systems method, ill-posed problems, monotone operators, iterative methods, fixed-point maps

feature in this paper is the justification of the DSM for non-smooth monotone operators, and the novel technique consists of the estimation of the derivative of the solutions to equations (2) and (7).

We make the following assumptions:

A) *B is a monotone, possibly nonlinear, hemicontinuous, defined on all of H operator in a real Hilbert space H.*

If A) holds, then the set $N := \{z : B(z) = 0\}$, if it is non-empty, is closed and convex, and therefore it has the unique element y with minimal norm.

Let $\dot{u}$ denote the derivative with respect to time. Consider the dynamical system (that is, the Cauchy problem):

$$\dot{w} = -B(w) - \epsilon w, \quad w(0) = w_0, \tag{2}$$

where w_0 is arbitrary.

The DSM in this paper consists of solving equation (1) by solving (2), and proving that for any initial approximation w_0 the following results (3) and (4) hold:

$$\exists w(t) \forall t > 0, \quad \exists V_\epsilon := w(\infty) := \lim_{t\to\infty} w(t), \quad B(V_\epsilon) + \epsilon V_\epsilon = 0, \tag{3}$$

and

$$\lim_{\epsilon\to 0} ||V_\epsilon - y|| = 0. \tag{4}$$

Conclusion (4) is known, but we give in subsection 2.5 a simple proof for convenience of the reader.

It is also known that equation (1) under the assumtions A) has a solution and this solution is unique. We prove this known fact by the new method, the DSM. If $\epsilon = 0$ then the limiting equation

$$B(u) = 0 \tag{5}$$

may have no solution (e.g., $B(u) = e^u$).We prove that if (5) has a solution, then the DSM allows one to construct a solution to (5). We assume

$$\epsilon(t) = \frac{c_1}{(c_0 + t)^b}, \quad c_0 > 0, \ c_1 > 0, \ 0 < b < 1, \tag{6}$$

where c_0, c_1 and b are constants. Consider the problem:

$$\dot{u} = -B(u) - \epsilon(t) u, \quad u(0) = u_0. \tag{7}$$

Our results are stated in two theorems:

Theorem 1. *If assumptions A) hold, equation (5) is solvable, and (6) holds, then problem (7) has a unique global solution $u(t)$, there exists strong*

limit $u(\infty) := \lim_{t\to\infty} u(t)$, $B(u(\infty)) = 0$, *and* $u(\infty) := y$ *is the unique minimal-norm element in the set of all solutions to (5).*

Theorem 2. *If assumptions A) hold and* $\epsilon = const > 0$, *then problem (2) has a unique global solution* $w(t)$, *there exists strong limit* $w(\infty)$, *and* $w(\infty)$ *solves (1).*

In Section 2 proofs are given.

2 Proofs

1. In Lemma 1 the existence of the unique global solutions to problems (2) and (7) is claimed. The proof of this Lemma will be given at the end of the paper to make the presentation self-contained. The result of Lemma 1 is known. Our proof is based on the Peano approximations (cf [1]).

Lemma 1. *If* $\epsilon = const > 0$ *and assumptions A) hold, then problem (2) has a unique global solution. If assumptions A) and (6) hold, then (7) has a unique global solution.*

2. **Proof of Theorem 2.** The proof consists of the following steps:

a) we prove:

$$\sup_{t\ge 0} ||u(t)|| < c < \infty; \quad g(t) \le g(0)e^{-\epsilon t}, \quad g(t) := ||w(t+h) - w(t)||, \tag{8}$$

where $c > 0$ stands for various estimation constants, and $h > 0$ is an arbitrary number.

This and the Cauchy test imply the existence of $V_\epsilon := w(\infty)$.

b) we prove that

$$||\dot{w}(t)|| \le ||\dot{w}(0)||e^{-\epsilon t}. \tag{9}$$

Thus, $\lim_{t\to\infty} ||\dot{w}(t)|| = 0$. Estimate (9) implies $\int_0^\infty ||\dot{w}(t)||dt < \infty$. This again, independently, implies the existence of $V_\epsilon := w(\infty)$.

From a), b), and from (2), one concludes that V_ϵ solves equation (1). To pass to the limit in (2) one uses demicontinuity of hemicontinuous, monotone, defined on all of H, operators, i.e., the property which says that $w \to V$ implies $B(w) \rightharpoonup B(V)$, (cf, e.g., [1], p.98). Here and below $\rightharpoonup$ denotes weak convergence in H.

The proof of (4) is given in subsection 2.5.

To complete the proof of Theorem 2, let us prove (8) and (9).

Let $z := w(t+h) - w(t)$ and $g := ||z||$. From equation (2) and from the monotonicity of B one gets:

$$g\dot{g} = -(B(w(t+h)) - B(w(t)) + \epsilon z, z) \le -\epsilon g^2.$$

Since $g \geq 0$, this implies the second half of (8). Its first half, namely the estimate $\sup_{t\geq 0} ||u(t)|| < c < \infty$, is proved below formula (16) for the more general case when ϵ depends on t.

Let $\psi := \frac{||z||}{h}$. Then, as above, one gets $\psi\dot{\psi} \leq -\epsilon\psi^2$. Thus, $\psi(t) \leq \psi(0)e^{-\epsilon t}$. Let $h \to 0$ and get (9). Theorem 2 is proved. $\Box$

3. **Proof of Theorem 1.** The scheme of the proof is similar to the one used above, but there are new points due to the dependence of $\epsilon(t)$ on t now. Denote $g(t) := ||u(t+h) - u(t)||$ and $z := u(t+h) - u(t)$. From (7) one gets

$$g\dot{g} = -(B(w(t+h)) - B(w(t)) + \epsilon(t)z, z) - (\epsilon(t+h) - \epsilon(t))(u(t+h), u(t)) \leq$$

$$-\epsilon g^2 + |\epsilon(t+h) - \epsilon(t)|||u(t+h)||g. \tag{10}$$

We prove below that

$$\sup_{t>0} ||u(t)|| \leq c < \infty, \quad c = const > 0. \tag{11}$$

From (10), (11) and (6) one gets the following differential inequality:

$$\dot{g} \leq -\epsilon(t)g + hc|\dot{\epsilon}(t)|, \tag{12}$$

where c is defined in (11).

From (12) one gets:

$$g(t) \leq e^{-\int_0^t \epsilon(s)ds}[g(0) + hc\int_0^t e^{\int_0^s \epsilon(x)dx}|\dot{\epsilon}(s)|ds]. \tag{13}$$

From (6) and (13) one gets

$$\lim_{t\to\infty} g(t) = 0, \quad \forall h > 0. \tag{14}$$

Indeed, if $a(t) := e^{\int_0^t \epsilon(s)ds}$, then $a^{-1}(t)\int_0^t a(s)|\dot{\epsilon}(s)|ds = O(\frac{1}{t})$ as $t \to \infty$, as one derives from assumption (6).

From (11) it follows that there exists a sequence $t_n \to \infty$, such that $u(t_n) \rightharpoonup v$, where $v \in H$ is some element. We prove below that $B(v) = 0$ by passing to the limit $t_n \to \infty$ in equation (7), using assumption (6), inequality (11), and relation (15), which one obtains dividing (14) by h and letting $h \to 0$:

$$\lim_{t\to\infty} ||\dot{u}(t)|| = 0. \tag{15}$$

Passing to the limit $t_n \to \infty$ in (7), proves that $u(\infty) := v$ solves (5).

Let us prove (11). In this proof we use the assumption that (5) has a solution y.

Denote $u(t) - y := p(t)$ and $||p|| := q$. Then

$$\dot{p} = -(B(u) - B(y)) - \epsilon(t)p - \epsilon(t)y. \tag{16}$$

Multiplying this by p and using the monotonicity of B, one gets:

$$\dot{q} \leq -\epsilon(t)q + \epsilon(t)||y||. \tag{17}$$

This implies $q(t) \leq a^{-1}(t)[q(0) + ||y|| \int_0^t a(s)\epsilon(s)ds]$. Thus,

$$||u(t) - y|| := q(t) \leq c, \tag{18}$$

so (11) follows (with a different c).

Let us now prove the existence of the strong limit $u(\infty)$ and the relation $u(\infty) = y$, where y is the unique minimal-norm solution to (5).

From (11) it follows that there is a sequence $t_n \to \infty$ such that $u(t_n) \rightharpoonup v$. From (6), (11), (15), and (7) one gets $\lim_{n\to\infty} B(u(t_n)) = 0$. This and assumptions A) imply $B(v) = 0$.

Let us prove that $u(t_n) \to v$. Since $u(t_n) \rightharpoonup v$, one gets

$$||v|| \leq \liminf_{n\to\infty} ||u(t_n)||.$$

If $\limsup_{n\to\infty} ||u(t_n)|| \leq ||v||$, then $\lim_{n\to\infty} ||u(t_n)|| = ||v||$, and together with the weak convergence $u(t_n) \rightharpoonup v$ this implies strong convergence $u(t_n) \to v$.

To prove that $\limsup_{n\to\infty} ||u(t_n)|| \leq ||v||$, we need some preparations. First, (6) implies that $\int_0^t \epsilon(s)ds \sim \frac{t^a}{a}$ as $t \to \infty$, where $a := 1 - b$, $0 < a < 1$. Second, (13) implies $||\dot{u}(t)|| \leq c/t$ as $t \to \infty$, where $c > 0$ is a constant. Indeed, if (6) holds, then $a^{-1}(t) \int_0^t a(s)|\dot{\epsilon}(s)|ds = O(\frac{1}{t})$ as $t \to \infty$. Equations $B(v) = 0$ and (7) imply $(B(u(t_n)) - B(v), u(t_n) - v) + \epsilon(t_n)(u(t_n), u(t_n) - v) = -(\dot{u}(t_n), u(t_n) - v)$. Since B is monotone, it follows that $(u(t_n), u(t_n) - v) \leq \frac{c}{t_n\epsilon(t_n)}$. Thus, $\limsup_{n\to\infty} ||u(t_n)|| \leq ||v||$, because $\lim_{n\to\infty} t_n\epsilon(t_n) = \infty$.

Let us prove that $v = y$, where y is the unique minimal-norm solution to (5). Replacing v by y in the above argument yields $(u(t_n), u(t_n) - y) \leq \frac{c}{t_n\epsilon(t_n)}$, so $||v|| = \limsup_{n\to\infty} ||u(t_n)|| \leq ||y||$. Since y is the unique minimal-norm solution to (5), and v solves (5), it follows that $v = y$.

Since the limit $\lim_{n\to\infty} u(t_n) = v = y$ is the same for every subsequence $t_n \to \infty$, for which the weak limit of $u(t_n)$ exists, one concludes that the strong limit $\lim_{t\to\infty} u(t) = y$. Indeed, assuming that for some sequence $t_n \to \infty$ the limit of $u(t_n)$ does not exist, one selects a subsequence, denoted again t_n, for which the weak limit of $u(t_n)$ does exist, and proves as before that this limit is y, thus getting a contradiction. Theorem 1 is proved. $\Box$

4. **Proof of Lemma 1.**

Let F be a nonlinear map satisfying assumptions A). In Lemma 1 this map is $F(u) := B(u) + \epsilon u$ for equation (2) and $F(u) := B(u) + \epsilon(t)u$ for equation (7). Our argument holds for any F satisfying assumptions A). We want to prove that the problem

$$\dot{w} = -F(w), \quad w(0) = w_0, \tag{19}$$

has a unique global solution.

Uniqueness of the solution is immediate: if w and v are solutions to (19), and $z := w-v$, then $\dot{z} = -[F(w)-F(v)]$, $z(0) = 0$. Multiplying by z and using the monotonicity of F, one gets $(\dot{z}, z) \leq 0$, so $||z(t)|| \leq 0$, and the uniqueness follows.

The proof of the global existence is less simple. To make the paper self-contained let us give a simplified version of the proof (cf [1]).

Consider the equation:

$$w_n(t) = w_0 - \int_0^t F(w_n(s - \frac{1}{n}))ds, \;\; t > 0; \quad w_n(t) = w_0, \;\; t \leq 0. \tag{20}$$

We wish to prove that

$$\lim_{n\to\infty} w_n(t) = w(t), \quad \forall t > 0, \tag{21}$$

where w solves (19). Recall that assumptions A) imply demicontinuity of F.

Fix an arbitrary $T > 0$, and let $B(w_0, r)$ be the ball centered at w_0 with radius $r > 0$. Let $\sup_{u\in B(w_0,r)} ||F(u)|| := c$. Then (20) implies $||w_n(t) - w_0|| \leq ct$. If $t \leq r/c$, then $w_n(t) \in B(w_0, r)$, and $||\dot{w}_n(t)|| \leq c$. Define

$$z_{nm}(t) := w_n(t) - w_m(t), \quad ||z_{nm}(t)|| := g_{nm}(t).$$

From (20) one gets:

$$g_{nm}\dot{g}_{nm} = -(F(w_n(t - \frac{1}{n})) - F(w_m(t - \frac{1}{m})), w_n(t) - w_m(t)) := I.$$

One has:

$$I = -(F(w_n(t - \frac{1}{n})) - F(w_m(t - \frac{1}{m})), w_n(t - \frac{1}{n}) - w_m(t - \frac{1}{m}))$$

$$-(F(w_n(t - \frac{1}{n})) - F(w_m(t - \frac{1}{m})), w_n(t) - w_n(t - \frac{1}{n}) - (w_m(t) - w_m(t - \frac{1}{m}))).$$

Using the monotonicity of F, the estimate $\sup_{w\in B(w_0,r)} ||F(w)|| \leq c$, and the estimate $||\dot{w}_n(t)|| \leq c$, one gets:

$$I \leq 4c^2(\frac{1}{n} + \frac{1}{m}).$$

Therefore

$$g_{nm}\dot{g}_{nm} \leq 4c^2(\frac{1}{n}+\frac{1}{m}) \to 0 \quad as \quad n,m\to\infty. \tag{22}$$

This implies

$$\lim_{n,m\to\infty} g_{nm}(t)=0, \quad 0\leq t\leq \frac{r}{c}. \tag{23}$$

Therefore there exists the strong limit $w(t)$:

$$\lim_{n\to\infty} w_n(t)=w(t), \quad 0\leq t\leq \frac{r}{c}. \tag{24}$$

The function w, defined in (24), satisfies the integral equation:

$$w(t)=w_0-\int_0^t F(w(s))ds, \tag{25}$$

and solves problem (19). If F is continuous, then problem (19) and equation (25) are equivalent. If F is demicontinuous, then they are also equivalent, but the derivative in (19) should be understood in the weak sense. We have proved the existence of the unique local solution to (19).

To prove that the solution to (19) exists for any $t\in[0,\infty)$, let us assume that the solution exists on $[0,T)$, but not on a larger interval $[0,T+d)$, $d>0$, and show that this leads to a contradiction. It is sufficient to prove that the finite limit:

$$\lim_{t\to T} w(t) \tag{26}$$

does exist, because then one can solve locally, on the interval $[T,T+d)$, equation (19) with the initial data $w(T)=\lim_{t\to T} w(t)$, and construct the solution to (19) on the interval $[0,T+d)$, thus getting a contradiction.

To prove that the finite limit (26) exists, consider

$$w(t+h)-w(t):=z(t), \quad ||z||:=g.$$

One has $\dot{z}=-[F(w(t+h))-F(w(t))]$. Using the monotonicity of F, one gets $(z,\dot{z})\leq 0$. Thus,

$$||w(t+h)-w(t)||\leq||w(h)-w(0)||. \tag{27}$$

The right-hand side in (27) tends to zero as $h\to 0$. This, and the Cauchy test imply the existence of the finite limit (26).

Lemma 1 is proved. $\Box$

5. **Proof of (4).**

This proof requires the following lemmas in which assumptions A) hold and are not repeated:

Lemma 2. *If y solves (5) and V_ϵ solves (3), then:*

$$||V_\epsilon|| \leq ||y||. \tag{28}$$

Lemma 3. *If $v_n \rightharpoonup v$ and $B(v_n) \to f$, then $B(v) = f$.*

Lemma 4. *If $v_n \rightharpoonup v$ and $||v_n|| \leq ||v||$, then $v_n \to v$.*

Assuming these lemmas, let us prove (4). From (28) one gets $V_\epsilon \rightharpoonup v$ (by a subsequence denoted again V_ϵ). Equation (3) implies $B(V_\epsilon) \to 0$. Thus, Lemma 3 yields $B(v) = 0$. From $V_\epsilon \rightharpoonup v$ and from (26) one gets

$$||v|| \leq \liminf_{\epsilon \to 0} ||V_\epsilon|| \leq \liminf_{\epsilon \to 0} ||V_\epsilon|| \leq ||y||.$$

Therefore $v = y$, since the solution to the equation (5), which has minimal norm, is unique, if A) holds. The weak convergence $V_\epsilon \rightharpoonup y$, inequality (28), and Lemma 4 imply (4).

Let us prove Lemmas 2-4.

Proof of Lemma 2. One has $B(V_\epsilon) + \epsilon V_\epsilon - B(y) = 0$. Multiply this by $V_\epsilon - y$ and use the monotonicity of B to get $\epsilon(V_\epsilon, V_\epsilon - y) \leq 0$. Since $\epsilon > 0$, inequality (28) follows. □

Proof of Lemma 3. The monotonicity of B implies

$$(B(v_n) - B(v - tz), v_n - v + tz) \geq 0 \quad \forall z \in H \ \forall t > 0.$$

Letting $n \to \infty$ one gets $(f - B(v - tz), tz) \geq 0$, so

$$(f - B(v - tz), z) \geq 0.$$

Letting $t \to 0$, one gets $(f - B(v), z) \geq 0 \ \forall z$. Thus, $B(v) = f$. Lemma 3 is proved. □

Proof of Lemma 4. One has $||v|| \leq \liminf_{n\to\infty} ||v_n|| \leq \limsup_{n\to\infty} ||v_n|| \leq ||v||$. Thus, $\lim_{n\to\infty} ||v_n|| = ||v||$. This and the weak convergence $v_n \rightharpoonup v$, imply: $||v - v_n||^2 = ||v_n||^2 + ||v||^2 - 2\Re(v_n, v) \to 0$. Lemma 4 is proved. □.

Proof of (4) is completed. □

References

[1] Deimling, K., Nonlinear functional analysis, Springer, Berlin, 1985

[2] Ramm, A.G., Dynamical systems method for solving operator equations, Comm. Nonlinear Sci. and Numer. Simul., 9, N2, (2004).